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Static-critical phenomenon analogy in absorptive optical bistability

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Abstract. The critical phenomenon analogy in an absorptive optical bistability system is studied. The asymptotic form of the family of potential functions for the system near threshold is obtained by using catastrophe theory. The well-known scaling hypothesis in the general homogeneous function form in critical phenomena is shown to be a characteristic of the asymptotic family. Four threshold exponents β, δ, γ, α and their four accompanied threshold amplitudes B, D, Γ, A on both bistability and monotonic regions are estimated. The threshold exponents obey the same scaling laws as those in critical phenomena, while the threshold amplitudes obey the definite relations between exponents and amplitudes.

1. Introduction

There is a close analogy between the phase transitions in equilibrium systems and the abrupt transitions of steady states in non-equilibrium systems [1, 2]. This analogy also occurs between the critical phenomena in equilibrium systems and the exponential behaviour of the abrupt transitions near thresholds, namely critical phenomenon analogy in non-equilibrium systems [2, 3].

Catastrophe theory (CT), first developed by Thom to understand discontinuous phenomena in nature, is a theory about topological stability and singularity for families of functions [4, 5]. When applied to discontinuous phenomena in any non-equilibrium system having a family of potential functions, it not only enables us to describe the abrupt transitions of steady states qualitatively, strictly speaking with diffeomorphic equivalent exactitude [4, 5], but also gives us the exact asymptotic form of the family of potential functions near threshold [3, 6]. CT is, therefore, powerful in discussing the critical phenomenon analogy in such non-equilibrium systems.

In the present paper, we have as our aim the critical phenomenon analogy in the system of absorptive optical bistability (AOB) [7]. The time evolution equation for AOB, which carries a family of potential functions and is suitable for CT analysis, is given in the following section. In section 3 the asymptotic form of the family of potential functions for the system near threshold $(x_c, C_c, y_c) = (\sqrt{3}, 4, 3\sqrt{3})$ is exactly derived. Then, the definitions of the threshold amplitudes and exponents are defined in parallel to the critical amplitudes and exponents in section 4, and the theoretical estimates of these amplitudes and exponents are made in section 5. Section 6 is devoted to the scaling relations obeyed by these threshold amplitudes and exponents and to the general homogeneous function characteristic of the asymptotic stationary potential of the AOB system near the threshold. Finally, in section 7, we draw some conclusions, making a brief comment on [8].

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2. Time evolution equation for the AOB system

In [7], the dynamics of AOB was studied in the semiclassical approximation by considering a mean-field quantum-mechanical model which consists of N $(N \gg 1)$ two-level atoms homogeneously filled in a pencil-shaped resonant cavity and driven by a coherent resonant incident field. Based on adiabatic elimination of the atomic variables the closed-time evolution equation for the AOB system reads

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \kappa[y - x - 2Cx/(1+x^2)] \tag{1}$$

where x and y are the saturation parameters proportional to the transmitted field E_T and incident field E_I, respectively, $x \geqslant 0$, $y \geqslant 0$, $C = g^2 N/2\kappa\gamma_\perp$ is the atomic cooperation parameter controlled by the atom-relevant and cavity geometric parameters such as the atom density and cavity length, $C \geqslant 0$, g is the coupling constant, κ is the cavity damping constant, $\kappa \ll \gamma_\perp, \gamma_\parallel$, and $\gamma_\perp$ and $\gamma_\parallel$ are, respectively, the transverse and longitudinal atomic relaxation rates.

Equation (1) can be rewritten in such a manner that a family of potential functions is explicitly introduced:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial G(x, C, y)}{\partial x} \qquad G(x, C, y) = \kappa[x^2/2 - yx + c\ln(1+x^2)]. \tag{2}$$

The family of potential functions, $G(x, C, y)$, characterizes the system. In particular, the stationary equation of the system is

$$\frac{\partial G(x, C, y)}{\partial x} = x - y + \frac{2xC}{1+x^2} = 0 \tag{3}$$

and the value of parameter x in a steady state for arbitrary fixed C and y is given by a solution of (3), which minimizes the potential of the system at (C, y).

The bistable behaviour in the sense that the parameter x (or the transmitted field E_T) is a discontinuous function of the parameter y (or the incident field E_I) with a hysteresis cycle was revealed. The bistability region of $C > 4$ and the monotonic region of $C < 4$ were predicted. One can find that the point $(x_c, C_c, y_c) = (\sqrt{3}, 4, 3\sqrt{3})$ is a threshold which distinguishes between the bistability and monotonic regions of the AOB system. Our interest is presently in the critical phenomenon analogy of the AOB system near this threshold.

Mathematically, $G(x, C, y)$ in the vicinity of the threshold (x_c, C_c, y_c) is an unfolding of the function $G(x, C_c, y_c)$ or, in our terminology, a family of potential functions, in which x plays the role of order parameter and C and y are two control parameters. We cannot simply treat $G(x, C, y)$ as a function of x only. To describe the critical phenomenon analogy, we have to find the asymptotic form of the family $G(x, C, y)$ near the threshold. CT is a good mathematical tool for this procedure.

3. Asymptotic form of the family of potential functions

For simplicity, we take the displacement transformations for the family $G(x, C, y)$,

$$x - x_c = m \qquad C - C_c = t \qquad y - y_c = h \tag{4}$$

where $(x_c, C_c, y_c) = (\sqrt{3}, 4, 3\sqrt{3})$, and let

$$G(m, t, h) \equiv G(x, C, y) - G(x_c, C_c, y_c)$$
$$= \kappa\{(m+x_c)^2/2 - x_c^2/2 - (m+x_c)(h+y_c) + x_c y_c + (t+C_c)\ln[1+(m+x_c)^2] - C_c \ln(1+x_c^2)\}. \tag{5}$$

$G(m, t, h)$ characterizes the AOB system as well as $G(x, C, y)$ and now $(m, t, h) = (0, 0, 0)$ is its isolated threshold.

The methods of calculation, notation and terms used below in this section can be found in books on catastrophe theory [4, 5] and in [6].

Letting $t = h = 0$ in $G(m, t, h)$, we have

$$j^4\{G(m, 0, 0)\} = \frac{\kappa}{16} m^4 \tag{6}$$

and hence

$$m^5 = j^5\left\{\left(\frac{16}{\kappa} m^2\right) j^3\left\{\frac{d}{dm} G(m, 0, 0)\right\}\right\} \tag{7a}$$

$$m^3 = j^3\left\{\left(\frac{16}{\kappa}\right) j^3\left\{\frac{d}{dm} G(m, 0, 0)\right\}\right\} \tag{7b}$$

where j^4, j^5 and j^3 are the truncation algebraic operators defined by

$$j^s\{f(x)\} = \sum_{r=0}^{s} \frac{1}{r!}\left(\frac{d^r}{dx^r} f(x)\right)\bigg|_0 x^r.$$

The equality (7*a*) implies that $G(m, 0, 0)$ is strongly 4-determinate, while (7*b*) implies that the codimension of $G(m, 0, 0)$ is 2. We can choose m and m^2 as the cobases generating a cospace of $G(m, 0, 0)$: $\{m, m^2\}$,

On the other hand, owing to (5), we can find

$$U_t(m) \equiv \frac{\partial}{\partial t} J^2\{G(m, t, 0)\} = \frac{\sqrt{3}}{2}\kappa m - \tfrac{1}{8}\kappa m^2 \tag{8}$$

$$U_h(m) \equiv \frac{\partial}{\partial h} J^2\{G(m, 0, h)\} = -\kappa m \tag{9}$$

at the point $(t, h) = (0, 0)$. J^2 in (8) and (9) is also a truncation algebraic operator,

$$J^s\{f(x)\} = \sum_{r=1}^{S} \frac{1}{r!}\left(\frac{d^r}{dx^r} f(x)\right)\bigg|_0 x^r$$

at $s = 2$, slightly different from the operator j^2.

$U_t(m)$ and $U_h(m)$ may expand a polynomial space, denoted by $\{U_t(m), U_h(m)\}$, which coincides with the cospace of $G(m, 0, 0)$, $\{m, m^2\}$. So the family $G(m, t, h)$ is universal according to the well-known Thom transversality theorem [4]. Furthermore, because $G(m, t, h)$ is universal and $M_1^3 \subseteq \Delta_3 G(m, 0, 0)$, where M_1^3 is a set of polynomials, and, because $G(m, 0, 0)$ is strongly 4-determinate, there must exist a neighbourhood of the threshold or the origin, $(m, t, h) = (0, 0, 0)$, in which the strong diffeomorphic equivalence

$$G(m, t, h) \sim j^5 G(m', 0, 0) + t' U_t(m') + h' U_h(m') \tag{10}$$

holds, where the strong diffeomorphic transformation

$$m' = m'(m, t, h) \qquad t' = t'(t, h) \qquad h' = h'(t, h) \tag{11}$$

does not change the origin and keeps the matrix

$$\begin{pmatrix} \frac{\partial m'}{\partial m} & \frac{\partial m'}{\partial t} & \frac{\partial m'}{\partial h} \\ \frac{\partial t'}{\partial m} & \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial h} \\ \frac{\partial h'}{\partial m} & \frac{\partial h'}{\partial t} & \frac{\partial h'}{\partial h} \end{pmatrix}_{(m,t,h)=(0,0,0)}$$

unitary.

As we are concerned only with the asymptotic behaviour of the AOB system near the threshold $(m, t, h) = (0, 0, 0)$, we can linearize this diffeomorphic transformation:

$$m' = m \qquad t' = t \qquad h' = h. \tag{12}$$

Putting (8) and (9) into (10), using (12) and dropping the higher infinitesimal terms as (m, t, h) is becoming close to $(0, 0, 0)$, we immediately obtain the asymptotic form of the family of potential functions near the threshold $(x_c, C_c, y_c) = (\sqrt{3}, 4, 3\sqrt{3})$:

$$\text{ASY } G(m, t, h) = \frac{\kappa}{16} m^4 + \frac{\sqrt{3}\,\kappa}{2} tm - \kappa hm. \tag{13}$$

There are only three terms in ASY $G(m, t, h)$. Two parameters t and h are independent and linearly coupled. Together with (13), the asymptotic time evolution equation of the AOB system in the vicinity of the threshold is

$$\frac{dm}{dt} = -\frac{\partial}{\partial m} \text{ASY } G(m, t, h). \tag{14}$$

4. Definitions of threshold amplitudes and exponents

The critical phenomenon analogy in non-equilibrium systems can be described by means of threshold amplitudes and exponents, as well as the critical phenomena in equilibrium systems by means of critical amplitudes and exponents. Here for the AOB system we shall be concerned with four static threshold exponents and their accompanying amplitudes. We define them in parallel to the usual critical amplitudes and exponents in equilibrium systems in order to facilitate comparison and to make them measurable.

Firstly, parameters m, t and h could each be negative or positive due to $x \geqslant 0$, $y \geqslant 0$, equations (4). Setting $y = 3\sqrt{3}$ in (3), we have $C(x) = (3\sqrt{3} - x)(1 + x^2)/2x$, $x \geqslant 0$ (AOB), and then we can find $C(\sqrt{3}) = 4$, $C'(\sqrt{3}) = 0$, $C''(\sqrt{3}) = 0$, $C'''(\sqrt{3}) < 0$. That indicates that $x < \sqrt{3}$ as $C > 4$ and $x > \sqrt{3}$ as $C < 4$ or, equivalently, $m < 0$ as $t > 0$ and $m > 0$ as $t < 0$. So, the consistent definitions of the threshold amplitudes B and B', and the threshold exponents β and β' are

$$m = \begin{cases} B(-t)^{\beta} & h = 0 \qquad t > 0 \\ B'(-t)^{\beta'} & h = 0 \qquad t < 0. \end{cases}$$

When $C = 4$, (3) yields $y = x + 8x/(1 + x^2)$, $x \geqslant 0$ (AOB). The inequality $y'(\sqrt{3}) > 0$ directly gives us that $y > 3\sqrt{3}$ as $x > \sqrt{3}$ and $y < 3\sqrt{3}$ as $x < \sqrt{3}$ or, equivalently, $h > 0$ as $m > 0$ and $h < 0$ as $m < 0$. This determines the manner by which the threshold amplitudes D and D', and the threshold exponents δ and δ' are defined:

$$h = \begin{cases} Dm^{\delta} & t = 0 \qquad h > 0 \\ D'm^{\delta'} & t = 0 \qquad h < 0 \end{cases}$$

In a similar way, we can consistently define other threshold amplitudes Γ, Γ', A, A' and threshold exponents γ, γ', α, α'. We list all the definitions of threshold amplitudes and exponents in table 1. For comparative purposes, the definitions of the critical amplitudes and exponents in the equilibrium magnetic system are listed correspondingly in the right-hand column of the table.

Table 1. The definitions of the threshold amplitudes and exponents. β, β', δ, δ', γ, γ', α, α' are the threshold exponents, while B, B', D, D', Γ, Γ', A, A' are the threshold amplitudes. They are defined in parallel to the usual critical amplitudes and exponents in the equilibrium magnetic system.

AOB system	Magnetic system [9]
m, order parameter	M, order parameter
t, control parameter relative to the atomic cooperative parameter C, $t = C - C_c$	t, reduced temperature, $t = (T - T_c)/T_c$
h, control parameter relative to the normalized incident field, $h = y - y_c$	H, applied magnetic field
S, formal entropy, $S = -\partial \mathrm{ASY}\ G/\partial t$	S, entropy, $S = -\partial G/\partial T$
$m = \begin{cases} B(-t)^{\beta} & h=0 \quad t>0 \\ B'(-t)^{\beta'} & h=0 \quad t<0 \end{cases}$	$M = B(-t)^{\beta} \quad H=0 \quad t<0$
$h = \begin{cases} Dm^{\delta} & t=0 \quad h>0 \\ D'm^{\delta'} & t=0 \quad h<0 \end{cases}$	$H = DM^{\delta} \quad t=0 \quad H>0$
$\chi = \frac{\partial m}{\partial h} = \begin{cases} \Gamma\lvert -t\rvert^{\gamma} & h=0 \quad t>0 \\ \Gamma'\lvert -t\rvert^{\gamma'} & h=0 \quad t<0 \end{cases}$	$\chi = \frac{\partial M}{\partial H} = \Gamma'(-t)^{\gamma'} \quad H=0 \quad t<0$
$c = \frac{\partial S}{\partial t} = \begin{cases} (A/\alpha)\lvert -t\rvert^{\alpha} & h=0 \quad t>0 \\ (A'/\alpha')\lvert -t\rvert^{\alpha'} & h=0 \quad t<0 \end{cases}$	$c = \frac{\partial S}{\partial t} = \frac{A'}{\alpha'}(-t)^{\alpha'} \quad H=0 \quad t<0.$

5. Estimates of threshold amplitudes and exponents

Due to (13) and (14), the asymptotic stationary equation of the AOG system near the threshold $(x_c, C_c, y_c) = (\sqrt{3}, 4, 3\sqrt{3})$ is

$$\tfrac{1}{4}m^3 + \frac{\sqrt{3}}{2}t - h = 0. \tag{15}$$

When $h = 0$, (15) has the solution

$$m = -(12)^{1/6}(t)^{1/3} \tag{16}$$

which implies

$$B = B' = (12)^{1/6} \qquad \text{and} \qquad \beta = \beta' = \tfrac{1}{3}.$$

Letting $t = 0$ in (15) immediately yields

$$h = \tfrac{1}{4}m^3 \tag{17}$$

which leads to

$$D = D' = \tfrac{1}{4} \qquad \text{and} \qquad \delta = \delta' = 3.$$

Table 2. The values of the threshold amplitudes and exponents. $\beta'=\beta$, $\delta'=\delta$, $\alpha'=\alpha$, $\gamma'=\gamma$, $B'=B$, $D'=D$, $A'=A$, $\Gamma'=\Gamma$. It is easy to see that the threshold exponents of the AOB system are beyond those of the Landau type. The AOB system could be a prototype of a new universality class in critical phenomena.

β	$\frac{1}{3}$
δ	3
γ	$\frac{2}{3}$
α	$\frac{2}{3}$
B	$(12)^{1/6}$
D	$\frac{1}{4}$
Γ	$\frac{4}{3}(12)^{-1/3}$
A	$(\kappa/18)(12)^{2/3}$

Next, taking a derivative on both sides of (15) with respect to the parameter h, and then letting $h=0$, we have

$$\left(\frac{\partial m}{\partial h}\right)_{h=0}=\tfrac{4}{3}m^2.$$

In addition, with (16) we have

$$\left(\frac{\partial m}{\partial h}\right)_{h=0}=\tfrac{4}{3}(12)^{-1/3}(-t)^{-2/3}.$$

Therefore,

$$\Gamma=\Gamma'=\tfrac{4}{3}(12)^{-1/3}\qquad\text{and}\qquad\gamma=\gamma'=\tfrac{2}{3}.$$

We take the second-order derivative of ASY $G(m, t, h)$ with respect to the parameter t and obtain

$$-\frac{\partial^2\,\mathrm{ASY}\;G(m,t,h)}{\partial t^2}=-\frac{3\kappa}{4}m^2\left(\frac{\partial m}{\partial t}\right)^2-\sqrt{3}\,\kappa\frac{\partial m}{\partial t} \tag{18}$$

where (15) is used for simplifying. On the other hand, the equation

$$\frac{\partial m}{\partial t}=-\frac{2}{\sqrt{3}\,m^2} \tag{19}$$

arises from taking a derivative of (15) with respect to the parameter t. Inserting (19) into (18) and using (16) we obtain

$$-\frac{\partial^2\,\mathrm{ASY}\;G(m,t,h)}{\partial t^2}=\kappa(12)^{-1/3}(-t)^{-2/3}$$

and hence

$$A=A'=\frac{\kappa}{18}(12)^{2/3}\qquad\text{and}\qquad\alpha=\alpha'=\tfrac{2}{3}.$$

The values of these threshold amplitudes and exponents are listed in table 2.

6. Scaling relations and scaling hypothesis

It is easy to see that the values of the threshold amplitudes and exponents obey the scaling relations

$$\alpha+2\beta+\gamma=2 \qquad \gamma=\beta(\delta-1)$$
$$\Gamma=\beta/DB^{\gamma/\beta} \qquad A=\kappa\alpha\beta DB^{\delta+1} \tag{20}$$

and another four relations of the same form (but with primes) for the threshold amplitudes and exponents.

The first pair of relations in (20) are the same as the scaling laws in critical phenomena of equilibrium systems. The remaining pair gives us the relations between amplitudes and exponents [3].

It is more interesting that the stationary potential of the AOB system is asymptotically a general homogeneous function (GHF) of its parameters t and h. In other words, the well-known scaling hypothesis in GHF form holds for the critical phenomenon analogy in the AOB system. To show this, let us recall the asymptotic family (13) and the asymptotic stationary equation (15). Suppose $m(t, h)$ is the value of order parameter m of the system in a steady state for small enough parameters (t, h), in other words $m(t, h)$ is a solution of (15) and minimizes the asymptotic potential at (t, h). This potential is

$$\begin{aligned} \mathrm{ASY}\ G(t, h) &\equiv \mathrm{ASY}\ G(m(t, h), t, h) \\ &= \frac{\kappa}{16} m^4(t, h) + \frac{\sqrt{3}\,\kappa}{2} t m(t, h) - \kappa h m(t, h). \end{aligned} \tag{21}$$

One can show that $\lambda^{1/4} m(t, h)$ satisfies

$$\tfrac{1}{4} m^3 + \frac{\sqrt{3}}{2} \lambda^p t - \lambda^q h = 0 \tag{22}$$

with $p = q = \frac{3}{4}$, where λ is an arbitrary positive number. In fact, substituting $\lambda^{1/4} m(t, h)$ into the left-hand side of (22), we have

$$\tfrac{1}{4}\lambda^{3/4} m^3(t, h) + \frac{\sqrt{3}}{2} \gamma^{3/4} t - \lambda^{3/4} h = \lambda^{3/4}\left(\tfrac{1}{4} m^3(t, h) + \frac{\sqrt{3}}{2} t - h\right) = 0.$$

The second equality is due to (15).

Equation (22) comes from ASY $G(m, \lambda^p t, \lambda^q h)$ as well as (15) from ASY $G(m, t, h)$. Therefore, replacing m in ASY $G(m, \lambda^p t, \lambda^q h)$ by $\lambda^{1/4} m(t, h)$, we can obtain the asymptotic stationary potential of the AOB system in a steady state at $(\lambda^p t, \lambda^q h)$:

$$\begin{aligned} \mathrm{ASY}\ G(\lambda^p t, \lambda^q h) &\equiv \mathrm{ASY}\ G(\lambda^{1/4} m(t, h), \lambda^p t, \lambda^q h) \\ &= \frac{\kappa}{16} \lambda m^4(t, h) + \frac{\sqrt{3}\,\kappa}{2} \lambda^{(p+1/4)} t m(t, h) - \kappa \lambda^{(q+1/4)} h m(t, h) \\ &= \lambda\ \mathrm{ASY}\ G(t, h). \end{aligned} \tag{23}$$

7. Concluding remarks

The AOB system is a typical open system far from equilibrium which exhibits the phase transition analogy. We have employed CT to investigate its critical phenomenon analogy. The values of the threshold amplitudes and exponents have been estimated. The values of the threshold exponents, $\beta' = \beta = \frac{1}{3}$, $\delta' = \delta = 3$, $\gamma' = \gamma = \frac{2}{3}$ and $\alpha' = \alpha = \frac{2}{3}$, are different from those of Landau's exponents ($\beta' = \beta = \frac{1}{2}$, $\delta' = \delta = 3$, $\gamma' = \gamma = 1$ and $\alpha' = \alpha = 0$). The AOB system could be a prototype of a new universality class in critical phenomena. Also, we have shown that the threshold amplitudes and exponents obey the same scaling laws as those in critical phenomena and that the well-known scaling hypothesis in GHF form still holds for the critical phenomenon analogy of the AOB system.

Our results support the close similarity between phase transitions in equilibrium systems and abrupt transitions of steady states in non-equilibrium systems.

Finally, we would like to make a brief comment on the work by Ou in [8]. Expanding $G(x, C, y)$ as if it was a function of only x, and taking a rotation transformation in the (C, y) parameter space, the author has artificially fitted the critical phenomenon analogy of the AOB system to Landau's type. But, in our opinion, the rotation transformation makes Ou's discussion lose explicit physical meaning from the experimental point of view and, from the theoretical point of view, it is unreasoning and there is no theoretical basis for treating $G(x, C, y)$ as a function of x only because $G(x, C, y)$ is an unfolding of the function $G(x, C_c, y_c)$ in the vicinity of the threshold. The real critical phenomenon analogy involved in the AOB system has thus been lost.

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